The above seminvariants may be converted into seminvariants of (A) by means of equations (5), (6), (7). A comparison of these equations with (9) shows that the desired seminvariants of (A) are obtained simply by replacing in the above semi-canonical forms π_{ikl} *by* u_{ikl}, π'_{ikl} *by* v_{ikl} and π''_{ikl} by w_{ikl}.

A comparison of the seminvariants $I^{(r\,s\,t)}$ of (A) with the corresponding seminvariants[5] for the case $n = 2$ shows the former to be independent. Moreover, the functional determinant of $I_l^{(r\,s\,t)}$ with respect to u_{ikl} for each value of $l = 0, 1, 2\ldots, m - 3$ shows that $I_l^{(r\,s\,t)}$ are independent among themselves and of the seminvariants $I^{(r\,s\,t)}$. Equations (10) show that we have the proper number of solutions for the variables involved and that all other seminvariants of the complete system can be obtained by the differentiation of $I^{(r\,s\,t)}$ and $I_l^{(r\,s\,t)}$. We have therefore the following theorem:

All seminvariants of (A) *are functions of* $I^{(r\,s\,t)}$ $(r = 0,1\ldots,n - 1; r + s \leq n;\ t = 1, 2, 3; t \leq s)$, $I_l^{(r\,s\,t)}$ $(r = 0,\ l\ldots,n - 1;\ r + s \leq n;\ t = 1, 2;\ t \leq s; l = 0,1,\ldots m - 3)$, *and of the derivatives of* $I^{(r\,s\,t)}$ $(t = 1,2)$ *and* $I_l^{(r\,s\,t)}$.

[1] Wilczynski, E. J., *Projective Differential Geometry of Curves and Ruled Surfaces*, Teubner, Leipzig, Chap. I.

[2] Wilczynski, E. J., *Ibid.*, Chap. II.

[3] Stouffer, E. B., *London, Proc. Math. Soc.* (Ser. 2), **15,** 1916 (217–226).

[4] Green, G. M., *Trans. Amer. Math, Soc., New York*, **16,** 1915 (1–12).

[5] Stouffer, E. B., *London, Proc. Math. Soc.* (Ser. 2), **17,** 1919 (337–352).

SOME NEW METHODS IN INTERIOR BALLISTICS*

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The principal problem of interior ballistics is, given a particular gun, a particular shot, and a particular kind of powder, to find, for a given load, the position and velocity of the shot, the mean pressure (and incidentally temperature) of the gases in the gun, and the fraction of the powder burned, all as functions of the time or of each other until the exit of the shot from the muzzle of the gun. In particular, we wish to know the muzzle-velocity of the shot, the maximum pressure to which the gun will be exposed, and the portion of the bore which will be exposed to it. It is then the duty of the mechanical engineer to design a gun to safely resist the pressure that may be expected. Or it is the inverse problem of the ballistician, by experiments on the action of the powder in the gun, to find its properties and those of the gun. It is true that the properties of the powder are more conveniently studied by means of

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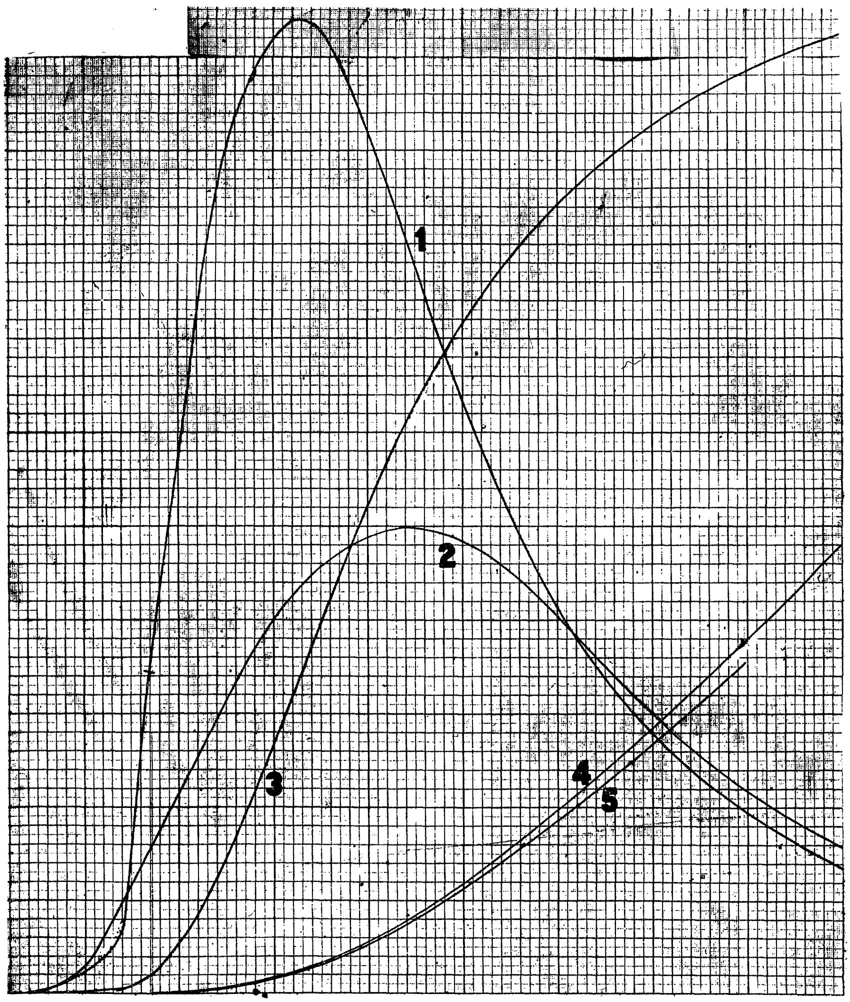
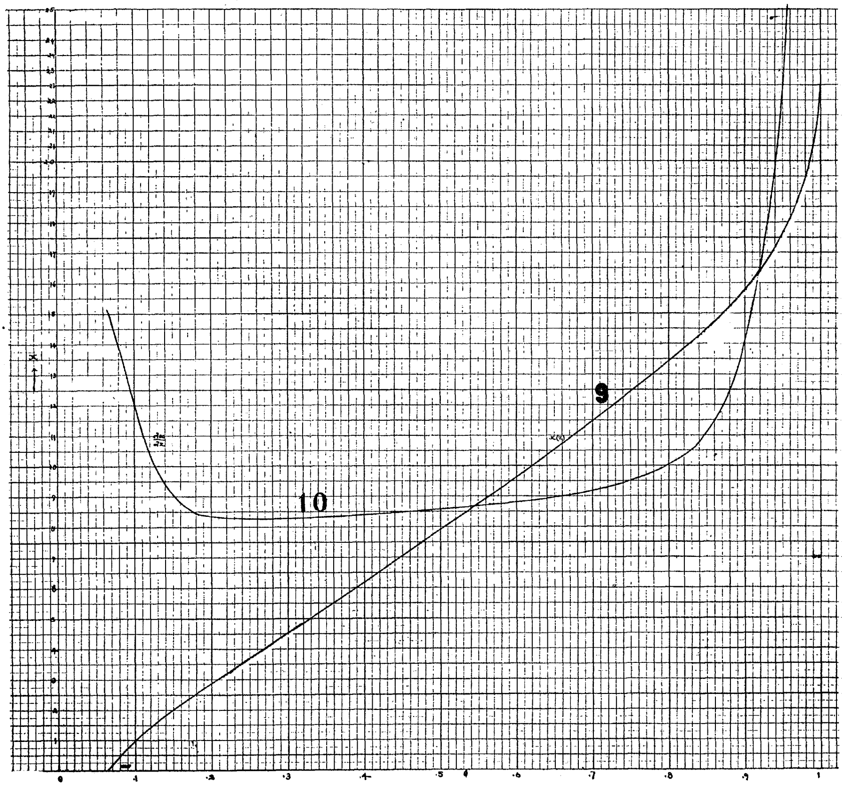
explosions in the bomb at constant volume, but we shall here show how all the properties may be determined by experiments in the gun alone.

It is customary among ballisticians to make many more or less crude hypotheses, and it is even denied that it is possible to apply the strict principles of thermodynamics to the gun as a heat-engine. It shall be our endeavor to show that this is not the case, and to relieve the subject



from some of the approximations that have usually been made. The latest and most distinguished writers upon this subject are General Charbonnier and his pupil Sugot, both members of the celebrated Commission de Gâvre (the latter its chief engineer) from which emanated all the knowledge of ballistics that was used during the war, not only for the French army but for ours.

The new methods here described consist, first, in the use of new experimental methods for obtaining the pressure in the gun, the position of the shot, and the muzzle velocity; second, in making the theory depend upon a new differential equation, and third, in the use of the graphic methods of calculation and integration that are common in engineering problems. The experimental methods hitherto have depended on the crusher gauge of Sir Andrew Noble and the Le Boulengé chronograph, both almost unchanged since their invention fifty years ago, and both giving but a single value of the pressure or the velocity. To be sure the Sébert veloci-



meter gives a continuous record of the position of the shot as a function of the time. It will readily be seen how a continuous record, with an infinity of values of the pressure, as well as a more direct measure of the positions of the shot, will increase our knowledge of the whole process.

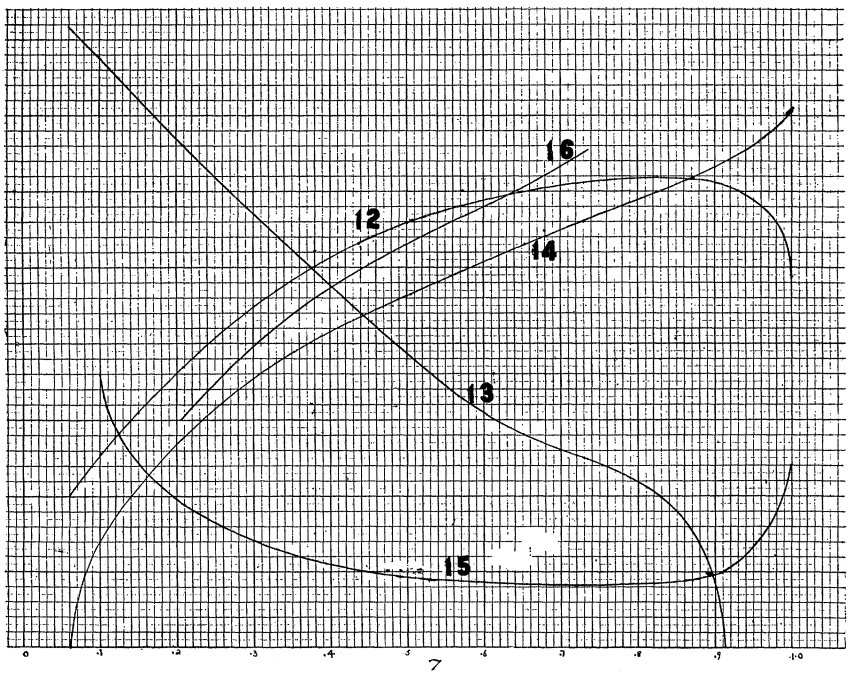
We must first decide upon the equation of state that is to be used for the gases generated by the explosion. It has been shown in a previous paper in these PROCEEDINGS (Vol. 5, pp. 286–288, July 1919) what equations of state are compatible with the constancy of the difference of specific heats at constant pressure and constant volume, and we shall use the equation of Clausius,

$$\left\{p + \frac{a}{T(v + \beta)^2}\right\}(v - \alpha) = RT, \tag{1}$$

in the simplified form, suitable for the high temperatures concerned (2000°–3000° C)

$$p(v - \eta) = RT \tag{2}$$

We shall use nearly the notation of Charbonnier and Sugot (*Cours de Balistique*, Imprimerie Nationale), but after having striven desperately with the multitude of engineers' units, pounds per square inch, kilograms per square centimeter or decimeter, inches or centimeters for bore and feet or meters for travel, and unknown units for velocity, acceleration and mass, we shall go back to first principle and express everything in C. G. S.



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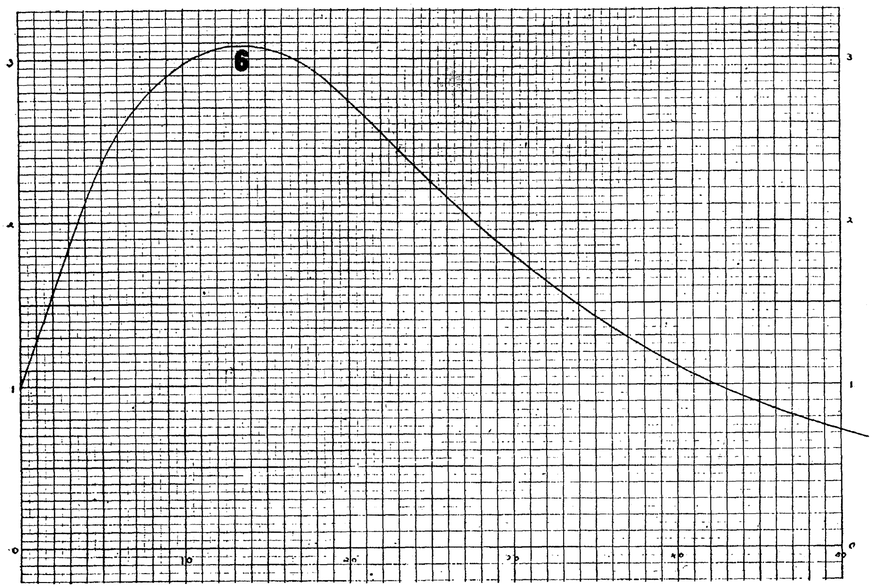
absolute units. It should be an axiom that no formula should contain any (arabic) numeral quantities except natural constants of the universe such as π, and that every formula should be correct in whatever system of units it is interpreted. It is hard to get engineers to agree to this. Consequently our unit of pressure shall be the dyne per square centimeter, with the convenience that a megadyne is between, and nearly equal to, the standard atmosphere and the kilogram per square centimeter at a place usually not specified, that v is the specific volume, in our units in cubic centimeters per gram or liters per kilogram, while T is the temperature on the absolute scale, in our units centigrade degrees. If we consider a load of powder of mass ω contained in a volume V, we define

the density of loading as $\Delta = \omega/V = 1/v$, while if the fraction of the load z has been converted into gas we have $v = V/\omega z$. Accordingly

$$p = \frac{RT}{v - \eta} = \frac{RT}{V/z - \eta} = \frac{RTz}{V - z\eta} \tag{3}$$

It is frequently assumed that during the burning the temperature of the gas remains constant, although this can hardly be the case during the expansion in the gun. Although the specific heats of the gases vary largely with the temperature, as shown by the classical experiments of Mallard and Le Chatelier (who found them to be linear functions of the temperature) we shall here consider them as constant, taking mean values. This hypothesis can be corrected later.



If the powder is contained in a variable volume c and ωz has been burned, while $\omega(1 - z)$ remains in solid form, of density δ, the gases will have the space $V = \left\{c - \omega(1 - z)/\delta\right\}$ to occupy, so that the specific volume will be $v = \left\{c - \frac{\omega}{\delta}(1 - z)\right\}/\omega z$ and we have

$$p = \frac{RT\omega z}{c - \omega(1 - z)/\delta - \omega z\eta} \tag{4}$$

as the relation between p, z, c, and T. Let S be the area of the bore, c' the volume of the chamber, and x the position or "travel" of the shot, so that

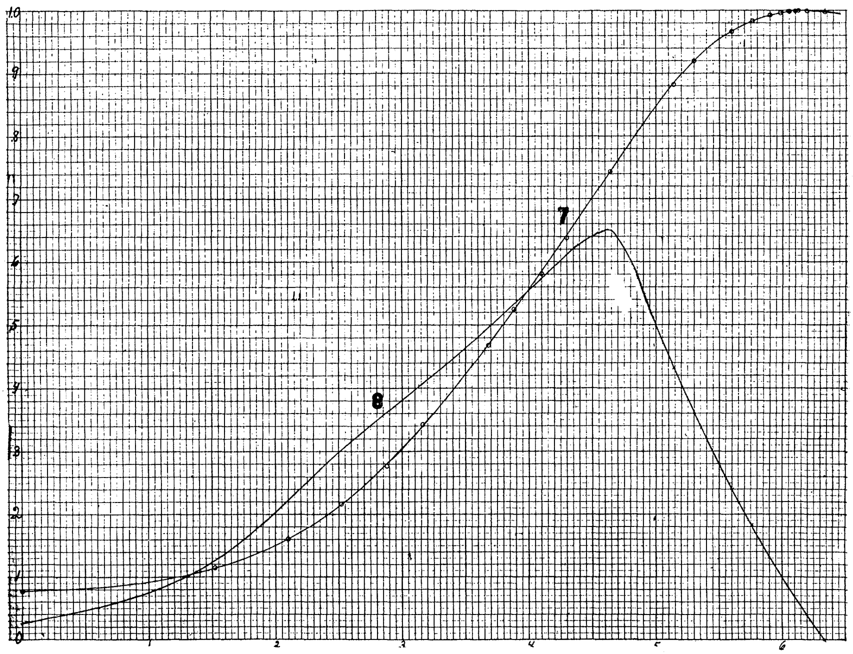
$$c = c' + Sx. \tag{5}$$

Then we have

$$p = \frac{RT\omega z}{c' + Sx - \omega(1 - z)/\delta - \omega z\eta}. \tag{6}$$

In order to write the dynamical equation of motion of the shot we shall assume that the resistance of the rifling (sometimes known as the passive resistance) which is due to the friction of the shot against the lands, and would naturally contain a term proportional to the pressure, is a linear function of the pressure

$$R = R_\circ + bp. \tag{7}$$



If then m is the mass of the shot, u its velocity, t the time, and if we take account of the inertia of the gas already formed, which is really ωz, by means of a factor λ, we have

$$\frac{dx}{dt} = u, \quad (8) \qquad (m + \lambda\omega)\frac{du}{dt} = S(p - p_\circ) - R_\circ - bp, \quad (9)$$

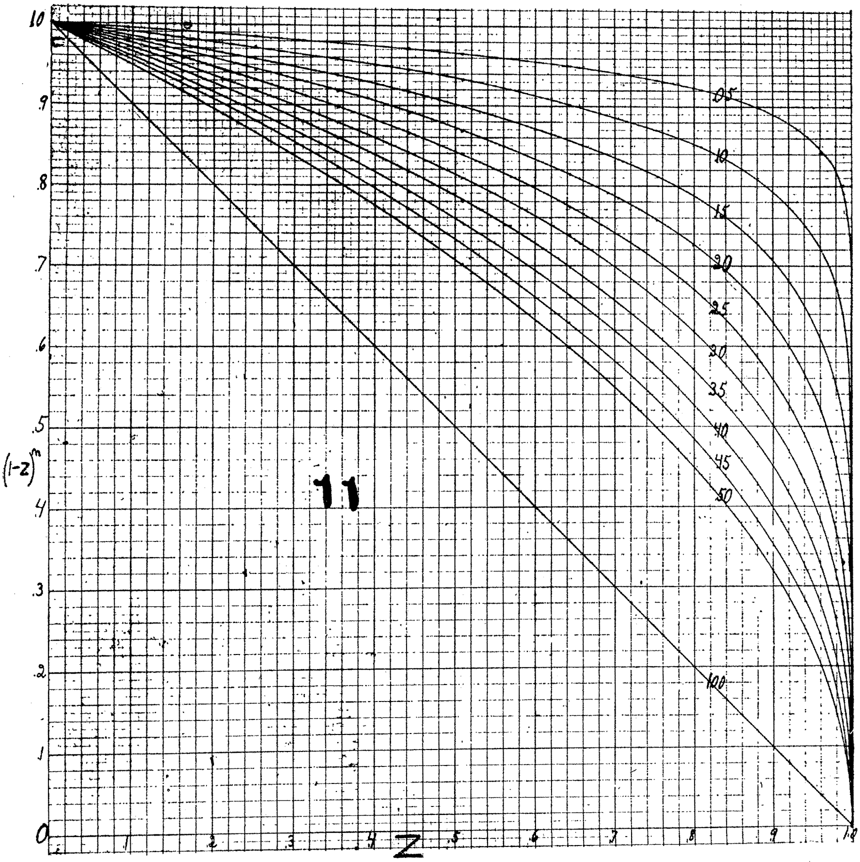
where $p_\circ$ is the atmospheric pressure, which is so small that it may generally be neglected.

By integration of (9) with respect to the time,

$$(m + \lambda\omega)u = (S - b)\int_0^t pdt - (Sp_\circ + R_\circ)t, \tag{10}$$

$$(m + \lambda\omega)x = (S - b)\int_0^t dt \int_0^t p dt - (Sp_\circ + R_\circ)t^2/2. \qquad (11)$$

If p is given in the form of a curve by the indicator, and the two integrals are evaluated for the whole time the shot is in the barrel, the determination of the muzzle velocity and of the whole length of the barrel, everything else being known, will furnish two linear equations for the determination of $R_\circ$ and b. On the other hand if the times at which the shot



passes a number of points are known, or if x is experimentally given as a function of t either of the equations (10) or (11) gives a means of verifying the assumption (7) and as many points as may be for determining the constants by least squares.

It will now be necessary to take account of the first law of thermodynamics. If U is the energy of the hot gas per unit of mass, dQ the amount of heat taken in, and Rdx the work done against friction, which is converted into heat, which stays in the gas, we have

$$dQ + Rdx = d(\omega z U) + pdV. \qquad (12)$$

If we now add (9) multiplied by (8) to (12) we have

$$dQ = (m + \lambda\omega)udu + d(\omega zU) + \frac{\omega}{\delta}pdz + Sp_{\delta}dx \qquad (13)$$

and if there are H units of heat evolved for each unit of mass of powder burned, and dq lost to the gun by conduction, we obtain the equation of conservation of energy,

$$H\omega dz - dq = (m + \lambda\omega)udu + d(\omega zU) + \frac{\omega}{\delta}pdz + Sp_{0}dx \quad (14)$$

Now if the specific heats are constant, we have

$$U = C_vT = \frac{RT}{\kappa - 1} = \frac{p(v - \eta)}{\kappa - 1} \qquad (15)$$

where κ is the ratio of the specific heats. The quantity of heat $Hdz - pdz/\delta$ may be determined by calorimetric experiments in a bomb at constant volume, as is seen by putting dx and du equal to zero. Putting this equal to $fdz/(\kappa - 1)$ as is customary, neglecting q and p_0, which can be done, and integrating (14) we have

$$\frac{f\omega z}{\kappa - 1} = (m + \lambda\omega)\frac{u^2}{2} + \frac{\omega zRT}{\kappa - 1} \qquad (16)$$

which is Résal's equation, given in 1864. Combining this with (6) we eliminate the unknown temperature, which has never been experimentally determined, and obtain

$$p = \frac{f\omega z - \frac{1}{2}(\kappa - 1)(m + \lambda\omega)u^2.}{c' + Sx - \omega(1 - z)/\delta - \omega\eta z} \qquad (17)$$

From this, if p, x, and u are experimentally determined, we may calculate z for each position of the shot.

If we could neglect u^2, or if it were proportional to z and if x, p, z, were taken as coördinates, this would be the equation of an hyperbolic paraboloid, of which the sections $z = const.$ are called *isopyric* lines by Lieut. Col. Hadcock in his paper on Internal Ballistics (*Proceedings Royal Society*, July 1, 1918 (479–509)). Col. Hadcock states that the expansion is neither adiabatic nor isothermal, but something between the two. This is true enough, as will be shown later, but it can hardly be justifiable to put, with any value of ϵ, Col. Hadcock's equation (5) $p(v - \alpha)^{\epsilon}/z^{\epsilon} = K$, for this makes the pressure proportional to z^{ϵ}, instead of a linear fractional function of z, The assumption that u^2/z is constant is not exactly true, but might be adopted without great error, as we shall see.

If we now put for brevity,

$$\frac{c'}{\omega} - \frac{1}{\delta} = B, \ \frac{S}{\omega} = C, \ \eta - \frac{1}{\delta} = D, \ \frac{S - b}{m + \lambda\omega} = E, \tag{18}$$

$$\frac{R_0 + Sp_0}{S - b} = F, \ \left(\frac{m}{\omega} + \lambda\right)\left(\frac{\kappa - 1}{2}\right) = G$$

where B and C are constants depending on the load, D a constant of the powder, E a constant of the gun and shot, and G a mixed mass and specific heat constant, or $g = f - Gu^2/z$ an approximate constant, we may write

$$p = \frac{fz - Gx^2}{B + Cx - Dz} \tag{19}$$

$$p = \frac{gz}{B + Cx - Dz} \tag{20}$$

We have heretofore said nothing about the rate of burning of the powder. After the powder is all consumed, $z = 1$, and the expansion is adiabatic. We are interested in the period of combustion. It is now customary to assume, after Charbonnier, that the rapidity of burning of the powder is a function of the amount already burned, the function $\varphi(z)$ being known as the form function of combustion, while a factor A is called the vivacity, being, for a given powder, inversely proportional to the linear dimensions of the grain of given form. The rate of burning is also proportional to some increasing function of the presssure, let us say $P(p)$. The simplest functions, except linear functions, are some powers of the variables, not in this case integral powers. If the power is progressive we take $\varphi(z) = z^\beta$, if degressive (as in our powders), $\varphi(z) = (1 - z)^\beta$. We will also put $P = p^\alpha$, where α is positive, as the burning is faster the greater the pressure. We accordingly put

$$\frac{dz}{dt} = A\varphi(z)P(p) = A(1 - z)^\beta p^\alpha. \tag{21}$$

We thus have the three simultaneous equations,

$$dt = \frac{dz}{A\varphi(z)P(p)} = \frac{dx}{u} = \frac{du}{E(p - F)}. \tag{22}$$

It is obvious that the simplest choice for independent variable is z, accordingly we take

$$\frac{dx}{dz} = \frac{u}{A\varphi(z)P(p)}, \tag{23}$$

$$\frac{du}{dz} = \frac{E\ (p - F)}{A\varphi(z)P(p)}, \tag{24}$$

and eliminating u by differentiation,

$$\frac{d^2x}{dz^2} = \frac{1}{A\varphi P}\frac{du}{dz} - \frac{u}{A\varphi^2P^2}\left(P\varphi' + \varphi P'\frac{dp}{dx}\right) =$$
$$\frac{E(p-F)}{A^2\varphi^2P^2} - \frac{dx}{dz}\left(\frac{P'}{P}\frac{dp}{dz} + \frac{\varphi'}{\varphi}\right). \quad (25)$$

Now by logarithmic differentiation of (19) we obtain

$$\frac{1}{p}\frac{dp}{dz} = \frac{f - 2Gu\dfrac{du}{dz}}{fz - Gu^2} - \frac{C\dfrac{dx}{dz} - D}{B + Cx - Dz} \quad (26)$$

or by means of (23), (24)

$$\frac{dp}{dz} = p\left\{\frac{f - 2GE(p-F)\dfrac{dx}{dz}}{fz - GA^2\varphi^2P^2\left(\dfrac{dx}{dz}\right)^2} - \frac{C\dfrac{dx}{dz} - D}{B + Cx - Dz}\right\}, \quad (27)$$

so that we have finally

$$\frac{d^2x}{dz^2} = \frac{E(p-F)}{A^2\varphi^2P^2} - \frac{dx}{dz}$$
$$\left[\frac{\varphi'}{\varphi} + p\frac{P'}{P}\left\{\frac{f - 2GE(p-E)\dfrac{dx}{dz}}{fz - GA^2\varphi^2P^2\left(\dfrac{dx}{dz}\right)^2} - \frac{C\dfrac{dx}{dz} - D}{B + Cx - Dz}\right\}\right] \quad (28)$$

This is a sufficiently complicated equation of the second order and of the third degree in dx/dz, but it is exact, and no simplifying assumptions have been made. If we know φ, P, p being obtained from equation (19), it gives x, the travel of the shot, in terms of z, the fraction of the powder burned. The equation (28) may be integrated graphically. When x is known as a function of z, p may be so determined, and then t and u from equations (22). Thus x, p, u, and z may all be found in terms of t, and the direct problem is solved. The equation (28) will be considerably simplified if we neglect G, which is equivalent to assuming the temperature constant, which is usually done.

We shall however use the differential equation (28) or rather (25) to solve the inverse problem. If x, p, and u are experimentally given, from which z is calculated by (19), everything in (28) is known except P and φ. If we begin with an approximate value of φ, say $(1 - z)^\beta$ and an assumed value of p^α, equation (28) will be a linear equation for α and β, from which by a few trials, the exact values may be obtained.

We now come to the experimental portion of the work. The pressures in the rifle were observed by means of the gauge described in these PROCEEDINGS for July, 1919, the film being afterwards placed in the lantern and an enlarged tracing being made with a pencil on squared paper. From

this the calculations were made by my assistant, Dr. E. A. Harrington, who has carried out all the experimental work except the determination of the travel of the shot. In the figures (p. 649), curve 1 represents p,t for the shot used, while 2 is a curve of a reduced charge of 7/10 normal. The curves were then graphically integrated twice by counting the squares. Curves 3 and 4 show the first and second integrals $\int pdt$ and $\int dt \int pdt$, respectively. The travel of the shot was determined by observing the time of contact of the bullet with the end of a coil of fine wire forced down the barrel to a measured point, an oscillograph of high frequency being used, and the time being observed on one of the revolving drums previously described. This was done about a year ago by my then assistant, Mr. H. C. Parker, and the result, shown in curve 5, is the average of a good many shots. The exact agreement of the two curves 4, 5 shows the propriety of the assumption about the resistance, and determines b as $(S-b)/S = 0.894$. In this case $R_0 = 0$. We can also neglect p_0 so that $F = 0$.

The muzzle velocity and the time of the shot leaving the barrel were determined by the method described in these PROCEEDINGS, April, 1920, developed in this laboratory by Thompson, Hickman, and Riffolt. The constants of the gun were as follows:

Volume of chamber, $c = 4.18$ cm.
Area of bore, $S = .456$ cm.

As for the constants of the powder, the density was determined by Dr. Harrington in the pyknometer as 1.58 gm./cm. No attempt was made to determine η experimentally, that being best done in the bomb, but the value $\eta = 0.95$ was assumed. The load was $\omega = 3.100$ gm. To determine f, the "force" of the powder (which has the dimensions of energy/mass) two methods were used, which agreed very well. If it is assumed that the point of inflexion on the observed pressure curve marks the end of combustion, $z = 1$, f is determined from equation (19). If, on the other hand, equation (19) is used with observed values of p, x, and u, it will give values of z increasing to a maximum, and then diminishing. This maximum should be equal to unity, and after a few trials a value of f can be found that makes it so. As stated, this method agrees with the other. We obtain $f = 1.087 \times 10^{10}$ cm.^{2}/sec.^{2}, corresponding very well with the values for the French powders and cordite. (The values given by Charbonnier and Sugot involve gravitation units.) We thus obtain

$$B = 0.716 \text{ cm.}^3/\text{gm.}, \qquad C = 0.147 \text{ cm.}^2/\text{gm.},$$
$$D = 0.32 \text{ cm.}^3/\text{gm.}, \qquad E = 0.394 \text{ cm.}^2/\text{gm.}$$

If we take $\kappa = 1.2$, and $\lambda = 0.2$, we have $G = 0.333$. Also we find that g (eq. 20) varies by about 16%, nearly as a linear function of x or z.

In order to show the functional relation (19) between the five quantities p, x, u, z, G, an "*abaque à glissement*" was constructed, so that by placing a transparent index on four of the variables the fifth could be found. This will be described elsewhere, but inasmuch as it was found to take longer to set the index than to set up the numbers on the arithmometer, of which we have several models, it was not used. After z was calculated, curves were drawn: $6(p,x)$, the indicator diagram (p. 652) for the gun; $7(z,t)$, $8(dz/dt,t)$, $9(x,z)$, $10(dx/dz,z)$, $11(1-z)^n$ for various values of n; $12(p,z)$, $13(dp/dz,z)$ (with a different zero of ordinates); $14(u,z)$; $15(du/dz,z)$; $16(dz/dt,z)$.

For graphical differentiation, a simple contrivance was used consisting of two metal rulers jointed together at one end. Near the other end of one was placed a plane mirror normal to the edge of the rule. By looking at the curve in this, and making its reflection join it without a corner, the direction of tangency could be found more accurately than by making the ruler tangent. The other ruler is then set parallel to the horizontal lines of the paper, and for a horizontal distance of ten or twenty centimeters the vertical distance between the rules read off, giving the trigonometrical tangent. Thus in a few minutes we may lay down the derivative curve with nearly three figure accuracy.

I have the following criticisms to make on the curves. After they had been given to the photographer it was discovered that the left-hand portions of (9) and (10) are incorrect, since dx/dz must vanish when x and u are zero. This is probably due to an error in choosing this point, which should be at the inflexion of the p curve, which should have been determined with the mirror directly on the film, instead of on the enlargement. When this is done, the pressure to start the shot is found to be the same as on curve 2, which (like 1) is so good that films of two different shots may be laid one over the other without being distinguishable. Making this correction will slightly displace curves (3) and (4). I realize that I have given hardly enough here to verify the theory, but as this paper is too long already, and as I am very shorthanded for assistance, I consider it best to publish it for the method, and to leave the verification, which should be by the construction of the observed curves from calculation, for another paper. By the courtesy of Rear-Admiral Ralph Earle, we have a Naval one-pounder gun, to which we hope to apply the methods, on which we hope to improve by experience. For a larger gun, where the wires might be blown out of the gun by the blast, we have another method of finding the velocity in the bore.

It should have been stated in our former paper, and is now stated here, that these experiments have been made possible by means of a generous grant from the Rumford Fund of the American Academy of Arts and Sciences, and to my colleagues on the Rumford Committee I hereby express my hearty thanks.